Solution to Problem Set 3

1. The function $F: [0, +\infty) \to \mathbb{R}$ is a power function on the intervals $[0, R)$ and $(R, +\infty)$ separately, so it is continuous on $[0, R)$ and on $(R, +\infty)$. At R, we have

$$\lim_{r \to R^-} F(r) = \lim_{r \to R^-} \frac{GMr}{R^3} = \frac{GMR}{R^3} = \frac{GM}{R^2},$$

$$\lim_{r \to R^+} F(r) = \lim_{r \to R^+} \frac{GM}{r^2} = \frac{GM}{R^2},$$

$$F(R) = \frac{GM}{R^2}.$$

Since $\lim_{r \to R} F(r) = \frac{GM}{R^2} = F(R)$, it follows that F is continuous at R too. Therefore F is a continuous function (throughout its domain $[0, +\infty)$).

2. On the intervals $(-\infty, 0)$ and $(0, +\infty)$, f is always continuous since it is a combination of well-defined elementary functions only. In order for f to be continuous at 0, we need to require that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0).$$

Now we have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(e^x + \frac{\sin 2x}{x} \right) = \lim_{x \to 0^-} e^x + 2 \cdot \lim_{x \to 0^-} \frac{\sin 2x}{2x} = e^0 + 2 \cdot 1 = 3,$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3 \sin x + ae^x) = 3 \cdot \lim_{x \to 0^+} \sin x + a \cdot \lim_{x \to 0^+} e^x = 3 \cdot 0 + a \cdot e^0 = a,$$

$$f(0) = 3 \sin 0 + ae^0 = a;$$

so we must have $a = 3$.

3. (a) First note that $\lim_{x \to \frac{\pi}{2}} \cos x = \cos \frac{\pi}{2} = 0$. Since $\tan$ is continuous at 0, we have

$$\lim_{x \to \frac{\pi}{2}} \tan(\cos x) = \tan \left(\lim_{x \to \frac{\pi}{2}} \cos x \right) = \tan 0 = 0.$$

Finally since $\sin$ is continuous at 0, it follows that

$$\lim_{x \to \frac{\pi}{2}} \sin(\tan(\cos x)) = \sin \left(\lim_{x \to \frac{\pi}{2}} \tan(\cos x) \right) = \sin 0 = 0.$$

(b) First note that

$$\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} = \lim_{x \to 1} \left(\frac{1 - \sqrt{x}}{1 - x} \cdot \frac{1 + \sqrt{x}}{1 + \sqrt{x}} \right) = \lim_{x \to 1} \frac{1 - x}{(1 - x)(1 + \sqrt{x})} = \lim_{x \to 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{1 + \sqrt{1}} = \frac{1}{2}.$$

Since $\arcsin$ is continuous at $\frac{1}{2}$, we have

$$\lim_{x \to 1} \arcsin \frac{1 - \sqrt{x}}{1 - x} = \arcsin \left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x} \right) = \arcsin \frac{1}{2} = \frac{\pi}{6}.$$

(c) First note that for every $x > 0$ we have $-1 \le \sin x \le 1$, so $-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$. Since $\lim_{x\to+\infty} -\frac{1}{x} = 0 = \lim_{x\to+\infty} \frac{1}{x}$, by Squeeze Theorem we have $\lim_{x\to+\infty} \frac{\sin x}{x} = 0$. Since the exponential function is continuous at 0, we have

$$\lim_{x\to+\infty} 2^{\frac{\sin x}{x}} = 2^{\lim_{x\to+\infty}\frac{\sin x}{x}} = 2^0 = 1.$$

(d) First note that for every $x \neq 0$ we have $-1 \le \cos\frac{1}{x} \le 1$, and so $1 - x^2 \le 1 + x^2\cos\frac{1}{x} \le 1 + x^2$. Since $\lim_{x\to 0}(1 - x^2) = 1$ and $\lim_{x\to 0}(1 + x^2) = 1$, according to Squeeze Theorem we have

$$\lim_{x\to 0}\left(1 + x^2\cos\frac{1}{x}\right) = 1.$$

Finally since $\ln$ is continuous at 1, we have

$$\lim_{x\to 0}\ln\left(1 + x^2\cos\frac{1}{x}\right) = \ln 1 = 0.$$

4. (a) First note that

$$\lim_{x\to+\infty}\frac{\sqrt{x+1}-\sqrt{x}}{2} = \lim_{x\to+\infty}\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\cdot\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\right)$$
$$= \lim_{x\to+\infty}\frac{(x+1)-x}{2\left(\sqrt{x+1}+\sqrt{x}\right)} = \lim_{x\to+\infty}\frac{1}{2\left(\sqrt{x+1}+\sqrt{x}\right)} = 0.$$

Since the function $\sin$ is **continuous** at 0, we have

$$\lim_{x\to+\infty}\sin\frac{\sqrt{x+1}-\sqrt{x}}{2} = \sin\left(\lim_{x\to+\infty}\frac{\sqrt{x+1}-\sqrt{x}}{2}\right) = \sin 0 = 0.$$

(b) For every $x \ge 0$, the sum-to-product formula gives

$$\sin\sqrt{x+1} - \sin\sqrt{x} = 2\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}.$$

Since $-2 \le 2\cos\frac{\sqrt{x+1}+\sqrt{x}}{2} \le 2$ for every $x \ge 0$, we have

$$-2\sin\frac{\sqrt{x+1}-\sqrt{x}}{2} \le 2\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\sin\frac{\sqrt{x+1}-\sqrt{x}}{2} \le 2\sin\frac{\sqrt{x+1}-\sqrt{x}}{2} \qquad \text{for every } x \ge 0$$

(why not $-2\left|\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\right| \le 2\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\sin\frac{\sqrt{x+1}-\sqrt{x}}{2} \le 2\left|\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\right|$?). Now according to (a) we have

$$\lim_{x\to+\infty} 2\sin\frac{\sqrt{x+1}-\sqrt{x}}{2} = 0 \qquad \text{and} \qquad \lim_{x\to+\infty} -2\sin\frac{\sqrt{x+1}-\sqrt{x}}{2} = 0,$$

so by Squeeze Theorem we have

$$\lim_{x\to+\infty}\left(\sin\sqrt{x+1} - \sin\sqrt{x}\right) = \lim_{x\to+\infty}\left(2\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\right) = 0.$$

5. (⇒) If $\lim_{x\to a} f(x) = 0$, then since the absolute value function is continuous, we have

$$\lim_{x\to a}|f(x)| = \left|\lim_{x\to a} f(x)\right| = |0| = 0.$$

(⇐) Suppose that $\lim_{x\to a}|f(x)| = 0$. Since

$$-|f(x)| \leq f(x) \leq |f(x)|$$

for every x near a and $\lim_{x\to a} -|f(x)| = \lim_{x\to a}|f(x)| = 0$, we have

$$\lim_{x\to a} f(x) = 0$$

by Squeeze Theorem. ∎

6. Since f is a polynomial, it is a continuous function. We check that

$$\begin{aligned} f(0) &= 0^n - a_{n-1}0^{n-1} - a_{n-2}0^{n-2} - \cdots - a_1 \cdot 0 - a_0 \\ &= -a_0 \\ &< 0 \end{aligned}$$

and

$$\begin{aligned} f(1) &= 1^n - a_{n-1}1^{n-1} - a_{n-2}1^{n-2} - \cdots - a_1 \cdot 1 - a_0 \\ &= 1 - (a_0 + a_1 + \cdots + a_{n-1}) \\ &> 0. \end{aligned}$$

Applying Intermediate Value Theorem to f on the bounded closed interval $[0, 1]$, we conclude that there exists $c \in (0, 1)$ such that $f(c) = 0$, i.e. f has a root $c \in (0, 1)$. ∎

7. Let $g: [0, 1] \to \mathbb{R}$ be the function defined by

$$g(x) = f(x) - f(x + 1).$$

Since f is continuous, g is also continuous. We check that

$$g(0) = f(0) - f(1)$$

and

$$g(1) = f(1) - f(2) = f(1) - f(0).$$

Now we claim that there exists $c \in [0, 1]$ such that $g(c) = 0$. We consider the following two cases:

(i) If $f(0) = f(1)$, then the number $c = 0$ verifies our claim.

(ii) If $f(0) \neq f(1)$, then

$$g(0)g(1) = [f(0) - f(1)][f(1) - f(0)] < 0,$$

so our claim follows by applying Intermediate Value Theorem to g on the bounded closed interval $[0, 1]$.

Therefore, there exists $c \in [0, 1]$ such that $f(c) = f(c + 1)$. ∎

8. (a) Let $f(x) = x^3 - 4x - 1$ so that the given equation is simply $f(x) = 0$. Since f is a polynomial, it is continuous on $\mathbb{R}$. Now we have

$$f(-1) = 2 > 0 \qquad \text{and} \qquad f(0) = -1 < 0.$$

Applying Intermediate Value Theorem to f on the bounded closed interval $[-1, 0]$, we conclude that there exists $c \in (-1, 0)$ such that $f(c) = 0$, i.e. there is a solution to the given equation in the interval $(-1, 0)$.

(b) Let $g(x) = \ln x - x + \sqrt{x}$ so that the given equation is simply $g(x) = 0$. g is a continuous function on the interval $(0, +\infty)$. Now we have

$$g(2) = \ln 2 - 2 + \sqrt{2} > 0 \qquad \text{and} \qquad g(3) = \ln 3 - 3 + \sqrt{3} < 0.$$

Applying Intermediate Value Theorem to g on the bounded closed interval $[2, 3]$, we conclude that there exists $c \in (2, 3)$ such that $g(c) = 0$, i.e. there is a solution to the given equation in the interval $(2, 3)$.

(c) Let $h(x) = \cos\sqrt{x} - e^x + 2$ so that the given equation is simply $h(x) = 0$. h is a continuous function on the interval $[0, +\infty)$. Now we have

$$h(0) = \cos\sqrt{0} - e^0 + 2 = 2 > 0 \qquad \text{and} \qquad h(1) = \cos\sqrt{1} - e^1 + 2 < 0.$$

Applying Intermediate Value Theorem to h on the bounded closed interval $[0, 1]$, we conclude that there exists $c \in (0, 1)$ such that $h(c) = 0$, i.e. there is a solution to the given equation in the interval $(0, 1)$.

9. For every $x \in \mathbb{R} \setminus \{1\}$, we have

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\dfrac{(x+h)+1}{(x+h)-1} - \dfrac{x+1}{x-1}}{h} = \lim_{h\to 0}\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{h(x-1)(x+h-1)} \\ &= \lim_{h\to 0}\frac{2x-2(x+h)}{h(x-1)(x+h-1)} = \lim_{h\to 0}\frac{-2}{(x-1)(x+h-1)} = \frac{-2}{(x-1)^2}.\end{aligned}$$

Therefore f is differentiable on $\mathbb{R} \setminus \{1\}$, and its derivative $f' \colon \mathbb{R} \setminus \{1\} \to \mathbb{R}$ is given by

$$f'(x) = \frac{-2}{(x-1)^2}.$$

10. For every $x \in \left(0, \frac{\pi}{2}\right)$, we have

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\csc\big(2(x+h)\big) - \csc 2x}{h} = \lim_{h\to 0}\frac{\dfrac{1}{\sin(2x+2h)} - \dfrac{1}{\sin 2x}}{h} \\ &= \lim_{h\to 0}\frac{\sin 2x - \sin(2x+2h)}{h\sin 2x\sin(2x+2h)} = \lim_{h\to 0}\frac{-2\sin h\cos(2x+h)}{h\sin 2x\sin(2x+2h)} \\ &= \left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left[\lim_{h\to 0}\frac{-2\cos(2x+h)}{\sin 2x\sin(2x+2h)}\right] \\ &= 1 \cdot \frac{-2\cos 2x}{\sin^2 2x} = -2\csc 2x\cot 2x.\end{aligned}$$

Therefore f is differentiable on $\left(0, \frac{\pi}{2}\right)$, and its derivative $f' \colon \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$ is given by

$$f'(x) = -2\csc 2x\cot 2x.$$

11. For every $x \in (-1, +\infty)$, we have

$$\lim_{h \to 0} \frac{\ln\big(1 + (x+h)\big) - \ln(1+x)}{h} = \lim_{h \to 0} \frac{1}{h} \ln\left(\frac{1+x+h}{1+x}\right) = \lim_{h \to 0} \frac{1}{h} \ln\left(1 + \frac{h}{1+x}\right) = \lim_{h \to 0} \ln\left(1 + \frac{h}{1+x}\right)^{\frac{1}{h}}$$

$$= \ln \lim_{h \to 0} \left(1 + \frac{h}{1+x}\right)^{\frac{1}{h}} \qquad \text{(since ln is continuous)}$$

$$= \ln\left[\lim_{h \to 0} \left(1 + \frac{h}{1+x}\right)^{\frac{1+x}{h}}\right]^{\frac{1}{1+x}} = \ln\left(e^{\frac{1}{1+x}}\right) = \frac{1}{1+x}.$$

This shows that

$$\frac{d}{dx} \ln(1+x) = \frac{1}{1+x} \qquad \text{for every } x \in (-1, +\infty).$$

12. (a) If $x > 0$, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2-2}{h} = \lim_{h \to 0} 0 = 0,$$

so $f'(x) = 0$; if $x < 0$, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1-1}{h} = \lim_{h \to 0} 0 = 0,$$

so $f'(x) = 0$ also. Therefore $f'(x) = 0$ for every $x \neq 0$.

(b) Since $f'(x) = 0$ for every $x \neq 0$, we have

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} 0 = 0.$$

(c) Consider the limit $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$. We have

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{1-2}{h} = \lim_{h \to 0^-} \frac{-1}{h} = +\infty$$

but

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{2-2}{h} = \lim_{h \to 0^+} 0 = 0.$$

Therefore the limit $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist, i.e. f is not differentiable at 0.

Alternative solution: Since

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 1 = 1 \qquad \text{but} \qquad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2 = 2,$$

the limit $\lim_{x \to 0} f(x)$ does not exist. Therefore f is discontinuous at 0, and in particular, f is not differentiable at 0.

13. (a) Suppose that f is even. Then by definition, we have

$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0}\frac{f(x-h)-f(x)}{h} \qquad (f \text{ is even})$$

$$= \lim_{t\to 0} -\frac{f(x+t)-f(x)}{t} \qquad (\text{Change of variable } t=-h)$$

$$= -f'(x)$$

for every $x \in \mathbb{R}$. Therefore f' is odd.

(b) Suppose that f is odd. Then by definition, we have

$$f'(-x) = \lim_{h\to 0}\frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0}\frac{-f(x-h)+f(x)}{h} \qquad (f \text{ is odd})$$

$$= \lim_{t\to 0}\frac{f(x+t)-f(x)}{t} \qquad (\text{Change of variable } t=-h)$$

$$= f'(x)$$

for every $x \in \mathbb{R}$. Therefore f' is even.

(c) Suppose that f is periodic, with a period T. Then by definition, we have

$$f'(x+T) = \lim_{h\to 0}\frac{f(x+T+h)-f(x+T)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad (T \text{ is a period of } f)$$

$$= f'(x)$$

for every $x \in \mathbb{R}$. Therefore f' is also periodic with a period T.

14. Since f is continuous at 0, we must have

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0).$$

Now

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}[b(\sin x+\cos x)+c] = b(\sin 0+\cos 0)+c = b+c,$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{2x}{1+e^{\frac{1}{x}}} = \frac{2(0)}{1+0} = 0,$$

$$f(0) = a,$$

so $a=0$ and $b+c=0$. Moreover, since f is differentiable at 0, we must have $f'_+(0) = f'_-(0)$. Now

$$f'_+(0) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^+}\frac{b(\sin x+\cos x)+\overbrace{c}^{=-b}-\overbrace{a}^{=0}}{x} = b\lim_{x\to 0^+}\frac{\sin x+\cos x-1}{x}$$

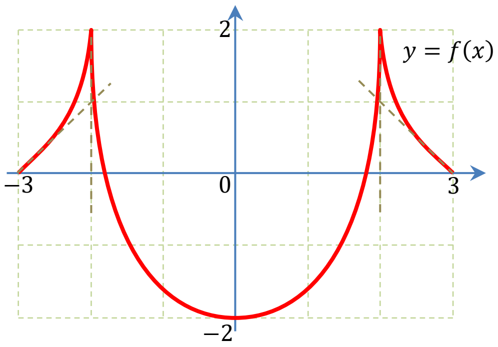
$$= b\lim_{x\to 0^+}\frac{\sin x-2\sin^2(x/2)}{x} = b\lim_{x\to 0^+}\left(\frac{\sin x}{x}-\frac{\sin(x/2)}{x/2}\sin\frac{x}{2}\right) = b\left(1-1\sin\frac{0}{2}\right) = b$$

and

$$f'_-(0) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^-}\frac{\frac{2x}{1+e^{1/x}}-0}{x} = \lim_{x\to 0^-}\frac{2}{1+e^{1/x}} = \frac{2}{1+0} = 2,$$

so we have $b=2$ and $c=-2$.

15.



16. The derivative of the given function f is

$$f'(x) = \frac{(5)(1+x^2) - (5x)(2x)}{(1+x^2)^2} = \frac{5(1-x^2)}{(1+x^2)^2}.$$

(a) Note that $f(2) = \frac{5 \cdot 2}{1+2^2} = 2$. At the point $\big(2, f(2)\big) = (2,2)$, the slope of the tangent line to the graph of f is $f'(2) = \frac{5(1-2^2)}{(1+2^2)^2} = -\frac{3}{5}$. Therefore the equation of the required tangent line is given by

$$y - 2 = -\frac{3}{5}(x-2),$$

i.e. $3x + 5y - 16 = 0$.

(b) Note that $f(3) = \frac{5 \cdot 3}{1+3^2} = \frac{3}{2}$. At the point $\big(3, f(3)\big) = \left(3, \frac{3}{2}\right)$, the slope of the tangent line to the graph of f is $f'(3) = \frac{5(1-3^2)}{(1+3^2)^2} = -\frac{2}{5}$, and so the slope of the normal line to the graph of f is $\frac{5}{2}$. Therefore the equation of the required normal line is given by

$$y - \frac{3}{2} = \frac{5}{2}(x-3),$$

i.e. $5x - 2y - 12 = 0$.

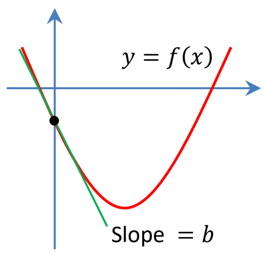
17. The derivative of the quadratic polynomial f is given by

$$f'(x) = 2ax + b.$$

At the point where the graph of f intersects the y-axis, i.e. when $x = 0$, we have

$$f'(0) = 2a(0) + b = b.$$

This shows that b is the slope of the tangent line to the parabola at the point where it intersects the y-axis. ■



18. (a)

$$f'(x) = \frac{(1013x^{1012})(1013x+1) - (x^{1013}+1)(1013)}{(1013x+1)^2}$$
$$= \frac{1013(1012x^{1013} + x^{1012} - 1)}{(1013x+1)^2}$$

(b)

$$f'(x) = \frac{(1)\left(x + \frac{2}{x}\right) - (x)\left(1 - \frac{2}{x^2}\right)}{\left(x + \frac{2}{x}\right)^2}$$
$$= \frac{4}{x\left(x + \frac{2}{x}\right)^2}$$

(c)

$$f'(x) = \frac{(1)(\sin x) - (x)(\cos x)}{(\sin x)^2} + 0$$

$\frac{\sin a}{a}$ is a real **constant**!

$$= \frac{\sin x - x\cos x}{\sin^2 x}$$
$$= \csc x - x\csc x\cot x$$

(d)

$$f'(x) = \frac{(1)(\sec x + \tan x) - (x)(\sec x\tan x + \sec^2 x)}{(\sec x + \tan x)^2}$$
$$= \frac{(\sec x + \tan x)(1 - x\sec x)}{(\sec x + \tan x)^2}$$
$$= \frac{1 - x\sec x}{\sec x + \tan x}$$